POISSON APPROXIMATION OF PROCESSES WITH LOCALLY INDEPENDENT INCREMENTS WITH MARKOV SWITCHING

V. S. Koroliuk [1], N. Limnios[2] and I.V. Samoilenko[1]

Abstract. In this paper, the weak convergence of additive functionals of processes with locally independent increments and with Markov switching in the scheme of Poisson approximation is proved. For the relative compactness, a method proposed by R. Liptser for semimartingales is used with a modification, where we apply a solution of a singular perturbation problem instead of an ergodic theorem.

1. Introduction

Poisson approximation is still an active area of research in several theoretical and applied directions. Several recent works on this topic can be found in the literature: we can find the classical approach in [1,2,3], and the functional approach in [8,9,7,12].

In particular in [8,9] it has been studied the following stochastic additive functional

$$\xi(t) = \xi_0 + \int_0^t \eta(ds; x(s)), \quad t \geq 0, \tag{1}$$

of a jump Markov process with locally independent increments (PLII) ([8, page 14]) $\eta(t; \cdot), t \geq 0$, (also known as a Piecewise deterministic Markov process – PDMP, [5, Chapter 2]), perturbed by the jump Markov process $x(t), t \geq 0$. The process (1) is studied in a (functional) Poisson approximation scheme, within an *ad hoc* time-scaling as we can see below (2).

In the Poisson approximation scheme, the jump values of the stochastic system are split into two parts: a small jump taking values with probabilities close to one and a big jump taken values with probabilities tending to zero together with the series parameter $\varepsilon \to 0$. So, in the Poisson approximation principle the probabilities (or intensities) of jumps are normalized by the series parameter $\varepsilon > 0$. Hence the time-scaled family of processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$, has to be considered.

However, the method used here to prove the weak convergence is quite different from the method proposed by other authors ([6]-[17]): the main point is to prove convergence of predictable characteristics of semimartingales which are integral functionals of some switching Markov processes. But the main difficulty is that the predictable characteristics of semimartingale themselves depend on the process we study. Thus, to prove the

1991 *Mathematics Subject Classification.* Primary 60J55, 60B10, 60F17, 60K10; Secondary 60G46, 60G60.

Key words and phrases. Poisson approximation, semimartingale, Markov process, locally independent increments process, piecewise deterministic Markov process, weak convergence, singular perturbation.

convergence of the process we should prove convergence of predictable characteristics that depend on the process. Ordinary methods can't help in this situation separately.

We propose to study functionals of PLII [8, page 14] using a combination of two methods. The method proposed by R. Liptser in [11], based on semimartingales theory, is combined with a solution of singular perturbation problem instead of ergodic theorem. So, the method includes two steps.

At the first step we prove the relative compactness of the semimartingale representation of the family ξ^ε, $\varepsilon > 0$, by proving the following two facts as proposed in Liptser [11]:

$$\lim_{c\to\infty} \sup_{\varepsilon\le\varepsilon_0} \mathbb{P}\{\sup_{t\le T} |\xi^\varepsilon(t)| > c\} = 0, \ \ \forall \varepsilon_0 > 0$$

that is known as compact containment condition (CCC), and

$$\mathbb{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le k|t-s|,$$

for some positive constant k.

At the second step we prove convergence of predictable characteristics of the semimartingales, which are integral functionals of the form ($a(u,x)$ is a real-valued function):

$$\int_0^t a(\xi^\varepsilon(s), x^\varepsilon(s))ds,$$

by using singular perturbation technique as presented in [8].

Finally, we apply Theorem IX.3.27 from [7] in order to prove the weak convergence of semimartingale.

The paper is organized as follows. In Section 2 we present the time-scaled additive functional (1), the PLII and the switching Markov process. In the same section we present the main results of Poisson approximation. In Section 3 we present the proof of the theorem.

2. Main results

Let us consider the space $\mathbb{R}^d$ endowed with a norm $|\cdot|$ ($d \ge 1$), and $(E, \mathcal{E})$, a *standard phase space*, (i.e., E is a Polish space and $\mathcal{E}$ its Borel σ-algebra). For a vector $v \in \mathbb{R}^d$ and a matrix $c \in \mathbb{R}^{d\times d}$, v^* and c^* denote their transpose respectively. Let $C_3(\mathbb{R}^d)$ be a measure-determining class of real-valued bounded functions g such that $g(u)/|u|^2 \to 0$, as $|u| \to 0$ (see [7,8]).

The additive functional $\xi^\varepsilon(t), t \ge 0, \varepsilon > 0$ on $\mathbb{R}^d$ in the series scheme with small series parameter $\varepsilon \to 0$, ($\varepsilon > 0$) is defined by the stochastic additive functional [8, Section 3.3.1]

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \int_0^t \eta^\varepsilon(ds; x(s/\varepsilon)). \tag{2}$$

The family of the Markov jump processes with *locally independent increments* $\eta^\varepsilon(t;x)$, $t \ge 0, x \in E$ on $\mathbb{R}^d$, is defined by the generators on the test-functions $\varphi(u) \in C^1(\mathbb{R}^d)$ [8, Section 3.3.1] (see also [9])

$$\widetilde{\Gamma}^\varepsilon(x)\varphi(u) = \varepsilon^{-1}\int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u)]\Gamma^\varepsilon(u, dv; x), \quad x \in E, \tag{3}$$

or, equivalently

$$\widetilde{\Gamma}^{\varepsilon}(x)\varphi(u) = b_{\varepsilon}(u;x)\varphi'(u) + \frac{1}{2}c_{\varepsilon}(u;x)\varphi''(u) + \varepsilon^{-1}\int_{\mathbb{R}^d}[\varphi(u+v) - \varphi(u) - v\varphi'(u)$$

$$-\frac{v^2}{2}\varphi''(u)]\Gamma^{\varepsilon}(u,dv;x),$$

where $b_{\varepsilon}(u;x) = \varepsilon^{-1}\int_{\mathbb{R}^d} v\Gamma^{\varepsilon}(u,dv;x)$, $c_{\varepsilon}(u;x) = \varepsilon^{-1}\int_{\mathbb{R}^d} vv^*\Gamma^{\varepsilon}(u,dv;x)$, and $\Gamma^{\varepsilon}(u,dv;x)$ is the intensity kernel.

The switching Markov process $x(t), t \geq 0$ on the standard phase space $(E,\mathcal{E})$, is defined by the generator

$$\mathbb{Q}\varphi(x) = q(x)\int_E P(x,dy)[\varphi(y) - \varphi(x)], \tag{4}$$

where $q(x), x \in E$, is the intensity of jumps function of $x(t), t \geq 0$, and $P(x,dy)$ the transition kernel of the embedded Markov chain $x_n, n \geq 0$, defined by $x_n = x(\tau_n), n \geq 0$, where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_n \leq ...$ are the jump times of $x(t), t \geq 0$. We suppose also that the processes $\eta^{\varepsilon}(t;x)$ and $x(t)$ are right continuous.

It is worth noticing that the coupled process $\xi^{\varepsilon}(t), x(t/\varepsilon), t \geq 0$ is a Markov additive process (see, e.g., [8, Section 2.5]).

The Poisson approximation of Markov additive process (2) is considered under the following conditions.

C1: The Markov process $x(t), t \geq 0$ is uniformly ergodic with $\pi(B), B \in \mathcal{E}$ as a stationary distribution.

C2: *Poisson approximation.* The family of processes with locally independent increments $\eta^{\varepsilon}(t;x), t \geq 0, x \in E$ satisfies the Poisson approximation conditions [8, Section 7.2.3]:

PA1: Approximation of the mean values:

$$b_{\varepsilon}(u;x) = \int_{\mathbb{R}^d} v\Gamma^{\varepsilon}(u,dv;x) = \varepsilon[b(u;x) + \theta_b^{\varepsilon}(u;x)],$$

and

$$c_{\varepsilon}(u;x) = \int_{\mathbb{R}^d} vv^*\Gamma^{\varepsilon}(u,dv;x) = \varepsilon[c(u;x) + \theta_c^{\varepsilon}(u;x)].$$

PA2: Poisson approximation condition for intensity kernel

$$\Gamma_g^{\varepsilon}(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma^{\varepsilon}(u,dv;x) = \varepsilon[\Gamma_g(u;x) + \theta_g^{\varepsilon}(u;x)]$$

for all $g \in C_3(\mathbb{R}^d)$, and the kernel $\Gamma_g(u;x)$ is bounded for each $g \in C_3(\mathbb{R}^d)$, that is,

$$|\Gamma_g(u;x)| \leq \Gamma_g \quad \text{(a constant depending on } g).$$

The above negligible terms $\theta_a^{\varepsilon}, \theta_b^{\varepsilon}, \theta_c^{\varepsilon}$ satisfy the condition

$$\sup_{x\in E}|\theta_{\cdot}^{\varepsilon}(u;x)| \to 0, \quad \varepsilon \to 0.$$

In addition the following conditions are used:

C3: *Uniform square-integrability*:

$$\lim_{c\to\infty}\sup_{x\in E}\int_{|v|>c} vv^*\Gamma(u,dv;x)=0,$$

where the kernel $\Gamma(u,dv;x)$ is defined on the class $C_3(\mathbb{R}^d)$ by the relation

$$\Gamma_g(u;x)=\int_{\mathbb{R}^d} g(v)\Gamma(u,dv;x),\quad g\in C_3(\mathbb{R}^d).$$

C4: *Linear growth*: there exists a positive constant L such that

$$|b(u;x)|\le L(1+|u|),\quad \text{and}\quad |c(u;x)|\le L(1+|u|^2),$$

and for any real-valued non-negative function $f(x), x\in E$, such that

$$\int_{\mathbb{R}^d\setminus\{0\}}(1+f(x))|x|^2dx<\infty,$$

we have

$$|\Lambda(u,v;x)|\le Lf(v)(1+|u|),$$

where $\Lambda(u,v;x)$ is the Radon-Nikodym derivative of $\Gamma(u,B;x)$ with respect to Lebesgue measure dv in $\mathbb{R}^d$, that is,

$$\Gamma(u,dv;x)=\Lambda(u,v;x)dv.$$

The main result of our work is the following.

Theorem 1. *Under conditions* **C1-C4** *the weak convergence*

$$\xi^\varepsilon(t)\Rightarrow\xi^0(t),\quad \varepsilon\to 0$$

takes place.

The limit process $\xi^0(t), t\ge 0$ is defined by the generator

$$\overline{\Gamma}\varphi(u)=\widehat{b}(u)\varphi'(u)+\int_{\mathbb{R}^d}[\varphi(u+v)-\varphi(u)-v\varphi'(u)]\widehat{\Gamma}(u,dv), \tag{5}$$

where the average deterministic drift is defined by

$$\widehat{b}(u)=\int_E\pi(dx)b(u;x),$$

and the average intensity kernel is defined by

$$\widehat{\Gamma}(u,dv)=\int_E\pi(dx)\Gamma(u,dv;x).$$

Remark 1. The limit process $\xi^0(t), t\ge 0$, is a PLII (see, e.g., [8, page 14]), (or a PDMP - see, e.g., [5, Chapter 2]). The generator (5) can be written also as follows

$$\overline{\Gamma}\varphi(u)=\widehat{b}_0(u)\varphi'(u)+\int_{\mathbb{R}^d}[\varphi(u+v)-\varphi(u)]\widehat{\Gamma}(u,dv),$$

where $\widehat{b}_0(u)=\widehat{b}(u)-\int_{\mathbb{R}^d}v\widehat{\Gamma}(u,dv)$.

In the following corollary of the above theorem we give an important particular case, where the limit process is a compound Poisson process.

Corollary 1. *Under Poisson approximation conditions:*
PA1': Approximation of mean values:

$$b_\varepsilon(u;x) = \int_{\mathbb{R}^d} v\Gamma^\varepsilon(dv;x) = \varepsilon[b(x) + \theta_b^\varepsilon(u;x)]$$

and

$$c_\varepsilon(u;x) = \int_{\mathbb{R}^d} vv^*\Gamma^\varepsilon(dv;x) = \varepsilon[c(x) + \theta_c^\varepsilon(u;x)].$$

PA2': Approximation condition for intensity kernel:

$$\Gamma_g^\varepsilon(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma^\varepsilon(u,dv;x) = \varepsilon[\Gamma_g(x) + \theta_g^\varepsilon(u;x)]$$

and the kernel $\Gamma_g(x)$ is bounded for each $g \in C_3(\mathbb{R}^d)$, that is,

$$|\Gamma_g(x)| \leq \Gamma_g \quad \textit{(a constant depending on g).}$$

And the additional condition
PA3:

$$\int_{\mathbb{R}^d} v\Gamma(dv) = \int_E \pi(dx)b(x), \quad \Gamma(dv) = \int_E \pi(dx)\Gamma(dv;x),$$

the limit process $\xi^0(t), t \geq 0$ is a compound Poisson process

$$\xi^0(t) = u + \sum_{k=1}^{\nu(t)} \alpha_k, \quad t \geq 0,$$

defined by the generator

$$\widetilde{\Gamma}\varphi(u) = \int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u)]\Gamma(dv),$$

where

$$\Gamma(dv) = \int_E \pi(dx)\Gamma(dv;x), \quad \Gamma_g(x) = \int_{\mathbb{R}^d} g(v)\Gamma(dv;x).$$

The sequence of random variables α_k, $k = 1, 2, \ldots$ is i.i.d. with joint distribution function $\mathbb{P}(\alpha_k \in dv) = \Gamma(dv)/\Lambda$, $\Lambda = \Gamma(\mathbb{R}^d)$ (it is obvious that $\Gamma(\mathbb{R}^d) = \int_{\mathbb{R}^d} \Gamma(dv)$). The time-homogeneous Poisson process $\nu(t), t \geq 0$, is defined by its intensity: $\Lambda > 0$.

3. Proof of Theorem 1

The proof of Theorem 1 is based on the semimartingale representation of the additive functional process (2). According to Theorems 6.27 and 7.16 [4] the predictable characteristics of the semimartingale (2) have the following representations:

- $B^\varepsilon(t) = \varepsilon^{-1}\int_0^t b_\varepsilon(\xi^\varepsilon(s); x_s^\varepsilon)ds = \int_0^t b(\xi^\varepsilon(s); x_s^\varepsilon)ds + \theta_b^\varepsilon$,
- $C^\varepsilon(t) = \varepsilon^{-1}\int_0^t c_\varepsilon(\xi^\varepsilon(s); x_s^\varepsilon)ds = \int_0^t c(\xi^\varepsilon(s); x_s^\varepsilon)ds + \theta_c^\varepsilon$,
- $\Gamma^\varepsilon(t) = \varepsilon^{-1}\int_0^t \int_{\mathbb{R}^d} g(v)\Gamma^\varepsilon(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds = \int_0^t \int_{\mathbb{R}^d} g(v)\Gamma(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds + \theta_g^\varepsilon$,

where $x_t^\varepsilon := x(t/\varepsilon), t \geq 0$, and $\sup\limits_{x \in E} |\theta_\cdot^\varepsilon| \to 0, \quad \varepsilon \to 0.$

The jump martingale part of the semimartingale (2) is represented as follows

$$\mu^\varepsilon(t) = \int_0^t \int_{\mathbb{R}^d} v[\mu^\varepsilon(\xi^\varepsilon(s), ds, dv; x_s^\varepsilon) - \Gamma^\varepsilon(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds].$$

Here $\mu^\varepsilon(u, ds, dv; x), x \in E$ is the family of counting measures with characteristics

$$\mathbb{E}\mu^\varepsilon(u, ds, dv; x) = \Gamma^\varepsilon(u, dv; x)ds.$$

We can see now that predictable characteristics depend on the process $\xi^\varepsilon(s)$. Thus, to prove convergence of $\xi^\varepsilon(s)$ we should prove convergence of predictable characteristics dependent on $\xi^\varepsilon(s)$. To avoid this difficulty, we combine two methods.

We split the proof of Theorem 1 in the following two steps.

STEP 1. At this step we establish the relative compactness of the family of processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$ by using the approach developed in [11]. Let us remind that the space of all probability measures defined on the standard space $(E, \mathcal{E})$ is also a Polish space; so the relative compactness and tightness are equivalent.

First we need the following lemma.

Lemma 1. *Under assumption* **C4** *there exists a constant $k_T > 0$, independent of ε and dependent on T, such that*

$$\mathbb{E} \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq k_T.$$

Proof: (following [11]). The semimartingale (2) has the following representation

$$\xi^\varepsilon(t) = u + A_t^\varepsilon + M_t^\varepsilon, \tag{6}$$

where $u = \xi^\varepsilon(0)$; A_t^ε is the predictable drift

$$A_t^\varepsilon = \int_0^t b(\xi^\varepsilon(s), x_s^\varepsilon)ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} v\Gamma(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds + \theta^\varepsilon,$$

and M_t^ε is the locally square integrable martingale

$$M_t^\varepsilon = \int_0^t c(\xi^\varepsilon(s); x_s^\varepsilon)dw_s + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} v[\mu^\varepsilon(ds, dv; x_s^\varepsilon) - \Gamma^\varepsilon(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds] + \theta^\varepsilon,$$

where $w_t, t \geq 0$ is a standard Wiener process.

For a process $y(t), t \geq 0$, let us define the process $y_t^\dagger = \sup_{s \leq t} |y(s)|$, then from (6) we have

$$((\xi_t^\varepsilon)^\dagger)^2 \leq 3[u^2 + ((A_t^\varepsilon)^\dagger)^2 + ((M_t^\varepsilon)^\dagger)^2]. \tag{7}$$

Condition **C4** implies that

$$(A_t^\varepsilon)^\dagger \leq L \int_0^t (1 + (\xi_s^\varepsilon)^\dagger)ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} |v| f(x)(1 + (\xi_s^\varepsilon)^\dagger)ds$$

$$\leq L(1+r_1)\int_0^t (1+(\xi_s^\varepsilon)^\dagger)ds, \tag{8}$$

where $r_1 = \int_{\mathbb{R}^d\setminus\{0\}} |x|^2 f(x)dx$.

Now, by Doob's inequality (see, e.g., [12, Theorem 1.9.2]),

$$\mathbb{E}((M_t^\varepsilon)^\dagger)^2 \leq 4|\mathbb{E}\langle M^\varepsilon\rangle_t|,$$

and condition **C4** we obtain

$$|\langle M^\varepsilon\rangle_t| = \left|\int_0^t c(\xi^\varepsilon(s); x_s^\varepsilon)c^*(\xi^\varepsilon(s); x_s^\varepsilon)ds + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} vv^*\Gamma^\varepsilon(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds + \theta^\varepsilon\right|$$

$$\leq 2L(1+r_1)\int_0^t [1+((\xi_s^\varepsilon)^\dagger)^2]ds. \tag{9}$$

Inequalities (7)-(9) and Cauchy-Bunyakovsky-Schwartz inequality, ($[\int_0^t \varphi(s)ds]^2 \leq t\int_0^t \varphi^2(s)ds$), imply

$$\mathbb{E}((\xi_t^\varepsilon)^\dagger)^2 \leq k_1 + k_2\int_0^t \mathbb{E}((\xi_s^\varepsilon)^\dagger)^2 ds,$$

where k_1 and k_2 are positive constants independent of ε.

By Gronwall inequality (see, e.g., [6, page 498]), we obtain

$$\mathbb{E}((\xi_t^\varepsilon)^\dagger)^2 \leq k_1 \exp(k_2 t).$$

Hence the lemma is proved.

Corollary 2. *Under assumption* **C4***, the following CCC holds:*

$$\lim_{c\to\infty} \sup_{\varepsilon\leq\varepsilon_0} \mathbb{P}\{\sup_{t\leq T} |\xi^\varepsilon(t)| > c\} = 0, \ \ \forall\varepsilon_0 > 0.$$

Proof: The proof of this corollary follows from Kolmogorov's inequality.

Remark 2. Another way to prove CCC is proposed in [8, Theorem 8.10] and used by other authors [6, 17]. They use the function $\varphi(u) = \sqrt{1+u^2}$ and prove corollary for $\varphi(\xi_t^\varepsilon)$ by applying the martingale characterization of the Markov process.

That can be easily proved, due to specific properties of $\varphi(u)$.

Lemma 2. *Under assumption* **C4** *there exists a constant $k > 0$, independent of ε such that*

$$\mathbb{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t-s|.$$

Proof: In the same manner with (7), we may write

$$|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq 2|A_t^\varepsilon - A_s^\varepsilon|^2 + 2|M_t^\varepsilon - M_s^\varepsilon|^2.$$

By using Doob's inequality, we obtain

$$\mathbb{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq 2\mathbb{E}\{|A_t^\varepsilon - A_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s|\}.$$

Now (8), (9), and assumption **C4** imply

$$|A_t^\varepsilon - A_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s| \le k_3[1 + ((\xi_T^\varepsilon)^\dagger)^2]|t-s|,$$

where k_3 is a positive constant independent of ε.

From the last inequality and Lemma 1 the desired conclusion emerges.

Thus from Corollary 2 and Lemma 2 immediately follows compactness of the family of processes $\xi^\varepsilon(t), t \ge 0, \varepsilon > 0$.

STEP 2. The next step of proof concerns the convergence of the predictable characteristics. To do that, we apply the results of Sections 3.2-3.3 in [8] and the following theorem. $C_0^2(\mathbb{R}^d \times E)$ is the space of real-valued twice continuously differentiable by the first argument functions, defined on $\mathbb{R}^d \times E$ and vanishing at infinity, and $C(\mathbb{R}^d \times E)$ is the space of real-valued continuous bounded functions defined on $\mathbb{R}^d \times E$.

Theorem 2. *(*[8, Theorem 6.3]*) Let the following conditions hold for a family of coupled Markov process $\xi^\varepsilon(t), x^\varepsilon(t), t \ge 0, \varepsilon > 0$:*

CD1: *There exists a family of test functions $\varphi^\varepsilon(u,x)$ in $C_0^2(\mathbb{R}^d \times E)$, such that*

$$\lim_{\varepsilon\to 0} \varphi^\varepsilon(u,x) = \varphi(u),$$

uniformly by u, x.

CD2: *The following convergence holds for the generator $\mathbb{L}^\varepsilon$ of a coupled Markov process $\xi^\varepsilon(t), x^\varepsilon(t), t \ge 0, \varepsilon > 0$*

$$\lim_{\varepsilon\to 0} \mathbb{L}^\varepsilon \varphi^\varepsilon(u,x) = \mathbb{L}\varphi(u),$$

uniformly by u, x. The family of functions $\mathbb{L}^\varepsilon\varphi^\varepsilon, \varepsilon > 0$ is uniformly bounded, both $\mathbb{L}\varphi(u)$ and $\mathbb{L}^\varepsilon\varphi^\varepsilon$ belong to $C(\mathbb{R}^d \times E)$.

CD3: *The quadratic characteristics of the martingales that characterize a coupled Markov process $\xi^\varepsilon(t), x^\varepsilon(t), t \ge 0, \varepsilon > 0$ have the representation $\langle\mu^\varepsilon\rangle_t = \int_0^t \zeta^\varepsilon(s)ds$, where the random functions $\zeta^\varepsilon, \varepsilon > 0$, satisfy the condition*

$$\sup_{0\le s\le T} \mathbb{E}|\zeta^\varepsilon(s)| \le c < +\infty.$$

CD4: *The convergence of the initial values holds and*

$$\sup_{\varepsilon>0} \mathbb{E}|\zeta^\varepsilon(0)| \le C < +\infty.$$

Then the weak convergence

$$\xi^\varepsilon(t) \Rightarrow \xi(t), \quad \varepsilon \to 0,$$

takes place.

We consider the three component Markov process $B^\varepsilon(t), \xi^\varepsilon(t), x_t^\varepsilon, t \ge 0$ which can be characterized by the martingale

$$\mu_t^\varepsilon = \varphi(B^\varepsilon(t), \xi^\varepsilon(t), x_t^\varepsilon) - \int_0^t \mathbb{L}^\varepsilon \varphi(B^\varepsilon(s), \xi^\varepsilon(s), x_t^\varepsilon)ds,$$

where its generator $\mathbb{L}^\varepsilon$ has the following representation [8]

$$\mathbb{L}^\varepsilon = \varepsilon^{-1}\mathbb{Q} + \widetilde{\Gamma}^\varepsilon + \mathbb{B}^\varepsilon, \tag{10}$$

with $\widetilde{\Gamma}^\varepsilon$ given by (3), $\mathbb{Q}$ given by (4), and $\mathbb{B}^\varepsilon(u;x)\varphi(v) = b_\varepsilon(u;x)\varphi'(v)$.

According to [8, Theorem 7.3], under the conditions **C1-C3** the limit generator for $\widetilde{\Gamma}^\varepsilon, \varepsilon \to 0$ has the form (5). However in order to prove the convergence of predictable characteristics, it is sufficient to study the action of the generator $\mathbb{L}^\varepsilon$ on test functions of two variables $\varphi(v,x)$.

Thus, it has the representation

$$\mathbb{L}^\varepsilon\varphi(v,x) = [\varepsilon^{-1}\mathbb{Q} + \mathbb{B}]\varphi(v,x). \tag{11}$$

The solution of the singular perturbation problem at the test functions $\varphi^\varepsilon(v,x) = \varphi(v) + \varepsilon\varphi_1(v,x)$ in the form $\mathbb{L}^\varepsilon\varphi^\varepsilon = \widehat{\mathbb{L}}\varphi + \theta^\varepsilon\varphi$ can be found in the same manner with Proposition 5.1 in [8]. That is

$$\widehat{\mathbb{L}} = \widehat{\mathbb{B}}, \tag{12}$$

where $\widehat{\mathbb{B}}\varphi(v) = \widehat{b}\varphi'(v)$.

Similar results can be proved for two other predictable characteristics.

Now Theorem 2 may be applied.

We see from (10) and (12) that the solution of singular perturbation problem for $\mathbb{L}^\varepsilon\varphi^\varepsilon(u,v;x)$ satisfies the conditions **CD1, CD2**. Condition **CD3** of this theorem implies that the quadratic characteristics of the martingale, corresponding to a coupled Markov process, is relatively compact. The same result follows from the CCC (see Corollary 2 and Lemma 2) by [7]. Thus, the condition **CD3** follows from the Corollary 2 and Lemma 2. As soon as $B^\varepsilon(0) = B^0(0), \xi^\varepsilon(0) = \xi^0(0)$ we see that the condition **CD4** is also satisfied. Thus, all the conditions of above Theorem 2 are satisfied, so the weak convergence $B^\varepsilon(t) \Rightarrow B^0(t)$ takes place.

By the same reasoning we can show the convergence of the processes $C^\varepsilon(t)$ and $\Gamma^\varepsilon(t)$.

The final step of the proof is achieved now by using Theorem IX.3.27 in [7]. Indeed all the conditions of this theorem are fulfilled.

As we have mentioned, the square integrability condition 3.24 follows from CCC (see [7]). The strong dominating hypothesis is true with the majoration functions presented in the Condition **C4**. Condition **C4** implies the condition of big jumps for the last predictable measure of Theorem IX.3.27 in [7]. Conditions iv and v of Theorem IX.3.27 [7] are obviously fulfilled.

The weak convergence of predictable characteristics is proved by solving the singularly perturbation problem for the generator (11).

The last condition (3.29) of Theorem IX.3.27 is also fulfilled due to CCC proved in Corollary 2 and Lemma 2. Thus, the weak convergence is true.

We can see now that the limit Markov process is characterized by the following predictable characteristics

$$B^0(t) = \int_0^t b(\xi^0(s))ds, \quad C^0(t) = \int_0^t c(\xi^0(s))ds, \quad \Gamma_g^0(t) = \int_0^t \Gamma_g(\xi^0(s))ds.$$

So, the limit Markov process $\xi^0(t)$ can be expressed by the generator (5).

Theorem 1 is proved.

Acknowledgements. The authors thank University of Bielefeld for hospitality and financial support by DFG project 436 UKR 113/80/04-07. We also thank the anonymous referee for useful comments and remarks.

References

1. Barbour A.D., Holst L., Janson S., *Poisson Approximation*, Oxford University Press, Oxford, 1992.
2. Barbour A. D., Chen L. H. Y., (Eds), *An introduction to Stein's method*, IMS Lecture Note Series **4** (2005), World Scientific Press, Singapore.
3. Barbour A. D., Chen L. H. Y., (Eds), *Stein's method and applications*, IMS Lecture Note Series **5** (2005), World Scientific Press, Singapore.
4. Çinlar E., Jacod J., Protter P., *Semimartingale and Markov processes*, Z. Wahrschein. verw. Gebiete **54** (1980), 161–219.
5. Davis M.H.A., *Markov Models and Optimization*, Chapman & Hall, 1993.
6. Ethier S.N., Kurtz T.G., *Markov Processes: Characterization and convergence*, J. Wiley, New York, 1986.
7. Jacod J., Shiryaev A.N., *Limit Theorems for Stochastic Processes*, Springer-Verlang, Berlin, 1987.
8. Koroliuk V.S., Limnios N., *Stochastic Systems in Merging Phase Space*, World Scientific, Singapore, 2005.
9. Korolyuk, V.S., Limnios, N., *Poisson approximation of increment processes with Markov switching*, Theor. Probab. Appl. **49** (2005), no. 4, 629–644.
10. Kushner H.J., *Weak Convergence Methods and Singular Perturbed Stochastic Control and Filtering Problems*, Birkhäuser, Boston, 1990.
11. Liptser R. Sh., *The Bogolubov averaging principle for semimartingales*, Proceedings of the Steklov Institute of Mathematics (1994), no. 4, 1-12.
12. Liptser R. Sh., Shiryayev A. N., *Theory of Martingales*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1989.
13. Stroock D.W., Varadhan S.R.S., *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin, 1979.
14. Skorokhod A.V., *Asymptotic Methods in the Theory of Stochastic Differential Equations*, AMS, Providence **78** (1989).
15. Skorokhod A.V., Hoppensteadt F.C., Salehi H., *Random Perturbation Method with Application in Science and Engineering*, Springer, 2002.
16. Silvestrov D. S., *Limit Theorems for Randomly Stopped Stochastic Processes*, Series: Probability and its Applications, Springer, 2004.
17. Sviridenko M.N., *Martingale approach to limit theorems for semi-Markov processes*, Theor. Probab. Appl. **34** (1986), no. 3, 540–545.

[1]Institute of Mathematics, Ukrainian National Academy of Science, Kiev, Ukraine
[2]Laboratoire de Mathématiques Appliquées, Université de Technologie de Compiègne, France

E-mail address: `isamoil@imath.kiev.ua`